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A Method of Numerical Solution of Cauchy-Type Singular Integral Equations with Generalized Kernels and Arbitrary Complex Singularities

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A numerical method is proposed for the approximate solution of a Cauchy-type singular integral equation (or an uncoupled system of such equations) of the first or the second kind and with a generalized kernel, in the sense that, besides the Cauchy singular part, the kernel has also a Fredholm part presenting strong singularities when both its variables tend to the same end-point of the integration interval. In this case any type of real or generally complex singularities in the unknown function of the integral equation may be present near the end-points of the integration interval. The method proposed consists simply in approximating the integrals in the integral equation by using an appropriate numerical integration rule with generally complex abscissas and weights, followed by the application of the resulting approximate equation at properly selected complex collocation points lying outside the integration interval. Although no proof of the convergence of the method seems possible, this method was seen to exhibit good convergence to the results expected in an example treated.

1. INTRODUCTION

The numerical solution of Cauchy-type singular integral equations, which will be called in the sequel simply singular integral equations, has become a subject of intensive research in recent years because of the frequent appearance of such equations in problems of mathematical physics. Among these problems we can mention the following: plane static elasticity problems (problems of finite media or infinite media with a hole [1], inclusion problems [2], crack problems [3], etc.), antiplane elasticity problems [4], elastic wave propagation problems [5, 6], elastic–perfectly plastic crack problems [7, 8], problems of flow of ideal and not ideal fluids [9, 10], antenna and other electromagnetic scattering problems [11] (the equations of these problems are not exactly one-dimensional singular integral equations but very similar to them), waveguide and surface wave scattering and diffraction problems [12–14], random rough surface scattering problems [15], etc. An account of most of the existing

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numerical techniques for the solution of a singular integral equation along a contour can be found in a monograph by Ivanov [16]. Besides, a powerful method for the numerical solution of such an equation along the interval $(-1, 1)$, based on the reduction of this equation to a Fredholm integral equation, was recently developed by Dow and Elliott [17, 18]. In the same references an account of the existing methods for the solution of singular integral equations along a part of the real axis can also be found.

On the other hand, more direct methods for the numerical solution of singular integral equations, based on the reduction of such an equation to a system of linear equations, after approximating the integrals through numerical integration rules and applying the integral equations at a number of appropriately selected points of the integration interval (collocation points), have been recently proposed. A review on these methods was written by the present authors [19]. A more detailed analysis of some of these methods is contained in Refs. [3, 20].

Furthermore, the present authors recently extended the above-mentioned method of numerical solution of singular integral equations to the following equation [21]:

$$Aw(x)\, g(x) + B \int_{-1}^{1} \frac{w(t)\, g(t)}{t - x}\, dt + \int_{-1}^{1} w(t)\, k(t, x)\, g(t)\, dt = f(x), \qquad -1 < x < 1, \tag{1}$$

where the constants A and B are not real but complex numbers and the functions $k(t, x)$ and $f(x)$ are known and regular along the interval $[-1, 1]$. In this case the weight function $w(t)$, which was intentionally separated from the unknown function, will be of the form [21]

$$w(t) = (1 - t)^{\alpha}(1 + t)^{\beta}, \qquad \operatorname{Re} \alpha, \operatorname{Re} \beta > -1, \tag{2}$$

where α and β result to be also complex numbers although their sum is an integer [21]. In this case Gaussian Jacobi-type numerical integration rules with generally complex abscissas and weights have been used for the approximation of the integrals in Eq. (1). Furthermore, the collocation points used for the derivation of the system of linear equations were determined as the roots of a complex polynomial [21].

The aim of this paper is to generalize the results of Refs. [19, 20] for the treatment of the general case of singular integral equations of the first or the second kind with nonconstant complex coefficients, generalized complex kernels, and complex weight functions as described in more detail in the next section.

2. The Integral Equation

Consider the following singular integral equation:

$$A(x)\, w(x)\, g(x) + B(x) \int_{a}^{b} \frac{w(t)\, g(t)}{t - x}\, dt + \int_{a}^{b} w(t)\, K(t, x)\, g(t)\, dt = f(x), \quad a < x < b, \tag{3}$$

valid along a finite or infinite part (a, b) of the real axis. In Eq. (3) $A(x)$ and $B(x)$ are assumed to be known generally complex functions, continuous along the interval $[a, b]$ and without singularities at the points a and b, and $f(x)$ is also a known generally complex function along (a, b) but it is permitted to present power, logarithmic, or other complicated-type singularities near a and b. Furthermore, the kernel $K(t, x)$ is assumed regular along (a, b), like a Fredholm kernel, but it is permitted to present strong singularities of order (-1) when $x = a$ or b and $t \to a$ or b, respectively. This behavior of the kernel $K(t, x)$ is the basis for calling Eq. (3) *a singular integral equation with a generalized kernel.*

With regard to the meaning of each one of the terms of Eq. (3), it is very difficult to give an interpretation of general validity. In principle, the right-hand-side function $f(x)$ is an a priori known function along the integration interval, like the loading function in plane elasticity problems. On the left-hand side of Eq. (3), the separation of the integral term into two terms was made just for a better presentation of the numerical technique; we may consider only one integral term with a kernel presenting a Cauchy-type singularity. Yet, a very rough interpretation of the two integral terms in Eq. (3) is that the first of them assumes the part of the boundary of the medium near $x = t$ as straight, whereas the second of them is due to the fact that, in reality, the boundary of the medium is, in general, curvilinear. This term also takes into account other boundaries of the same medium (or interfaces), along which other singular integral equations hold. With regard to the free term of the left-hand side of Eq. (3), its appearance, whenever this term exists, may be interpreted by the fact that in physical problems we have limiting values of analytic functions as we approach a boundary and not values of these functions on the boundary itself. Next, these limiting values are expressed in terms of appropriately defined boundary values of these functions, which are denoted by Cauchy-type integrals, as well as the densities of these integrals, which constitute the free terms in Eq. (3). For example, in the theory of analytic functions, the well-known Plemelj formulas [22–24] are valid. Of course, all these comments are not strictly valid but may serve only for an illustration of the form of Eq. (3).

Finally, the unknown function $w(t)\, g(t)$ has been separated into a weight function $w(t)$, containing all singularities of the unknown function near a and b (like power or logarithmic singularities), and a regular part $g(t)$, which remains bounded along the closed interval $[a, b]$. With regard to the weight function $w(t)$, it will be assumed to be known in advance, that is, the singularities in the solution of Eq. (3) should be known before the numerical solution of this equation. But this does not cause any difficulty. In fact, there are two completely different methods for determining the singularities in the solution of Eq. (3), which should be exhibited by the weight function $w(t)$.

The first of these methods is based on physical considerations: near a singular point in a physical problem, like the end-points of the integration interval or the points of discontinuity of the right-hand-side function $f(x)$ in Eq. (3), the functions constituting the solution of a physical problem (eigher real or complex, but defined in a plane region surrounding the integration interval) should behave in some concrete way. This way can be determined if the fundamental equations of the problem under consideration are known together with the geometry and boundary conditions. Next, it is easy to determine the singularities in the weight function $w(t)$ in Eq. (3), which generally are the same as the previously mentioned singularities in the two-dimensional functions of the problem under consideration. This method for determining the singularities in $w(t)$ has the advantage that it does not require one to take into account Eq. (3) itself. Thus our attention is focused just near the singular points, where singularities are present, and the determination of the singularities, one by one, near each singular point, is easy to be made. This method is generally used in plane elasticity problems. An account of the corresponding techniques for determining singularities near singular points in plane elasticity can be found in Refs. [25, 26]. Moreover, to the authors' surprise, singularities completely analogous to those present in plane elasticity problems are also present in the problems of static equilibrium of liquid crystals of nematic type [27].

The second method for determining the singularities in the solution of Eq. (3) and, thus, the weight function $w(t)$ too, consists in taking into account this equation without paying any attention to the physical problem from which it was derived. This is completely possible if the values of $A(x)$ and $B(x)$ and the behaviors of $K(t, x)$ and $f(x)$ for t, x tending to a singular point (an end-point of the integration interval or a point of discontinuity of the functions entering Eq. (3)) are completely known and the theoretical results of Muskhelishvili and others [22–24] for the behavior of a Cauchy-type integral near a singular point are taken into consideration. Thus, if, for example, $f(x)$ in Eq. (3) remains finite near a singular point $x = c$, then $w(t)$ may become unbounded as $(t - c)^\lambda$ near this point and λ can be determined by taking into account Eq. (3) and the behavior of the integrals containing $w(t)$. A transcendental equation for the determination of λ results. The value, or rather values, of λ may result in a real or complex number. In general, there exists some value of λ with a real part less than 0 and greater than -1. Analogous arguments hold also for the case of logarithmic singularities. A more detailed analysis of this technique can be found in Refs. [28–29]. In spite of the general character of this method, it is not preferable to the first method for determining these singularities, mentioned in the previous paragraph, but can be used as a check of its results. The disadvantage of this second method is that it requires more algebraic manipulations for the derivation of the transcendental equation for the determination of the eigenvalues λ than the first method. Moreover, it can be mentioned that, in general, the weight function $w(t)$ will be complex along the integration interval (a, b) and will present complex power or logarithmic or other complicated singularities near a and b. Of course, it is not necessary that all the singular behaviors assumed for the functions entering Eq. (3) hold at the same time.

One final remark on the power singularities λ, incorporated in the weight function $w(t)$, should be made: In general, near a singular point there exists a series of eigenvalues λ, resulting from the solution of the above-mentioned transcendental equation. In practice, we take into account only the dominant singularity, corresponding to the eigenvalue λ with the least real part (in general between -1 and 0). This is almost always the case with real singularities and, sometimes, with complex singularities too (as happens in the plane elasticity problems of a crack along a straight or curvilinear interface or a rigid stamp acting on the straight boundary of an elastic half-plane in the case of nonnegligible friction). Unfortunately, in other applications it is possible to have a pair of two complex conjugate singularities instead of one dominant singularity. In this case both these singularities should be taken into account and the weight function $w(t)$ behaves as $[(t-c)^{\lambda} + C(t-c)^{\bar{\lambda}}]$ near a singular point $t = c$. Fortunately, at least in plane elasticity problems, it is possible to determine the value of the constant C and thus to completely determine the behavior of $w(t)$ near $t = c$ [26, 30].

Before proceeding to our analysis, we would like to mention that:

(i) Equation (3) is not reducible to a Fredholm integral equation. In fact, it results when we have to solve a singular integral equation along a nonsmooth curve [24]. Better, in most cases a system of uncoupled singular integral equations of form (3) results, but the method of numerical solution remains unaltered in the case of systems of uncoupled singular integral equations of form (3). Furthermore, the theoretical investigations on the solutions of singular integral equations or systems of such equations contained in the well-known monographs of Muskhelishvili [22], Gakhov [23], and Vekua [31] do not apply to Eq. (3) because of the generalized character of its kernel $K(t, x)$ in the sense mentioned previously. Moreover, the concept of the index of a singular integral equation is of very little help for equations of form (3). Also the dominant equation of Eq. (3) cannot reveal the behavior of the unknown function at the end-points of the integration interval.

(ii) All these difficulties have made the numerical solution of Eq. (3) almost impossible. To the authors' knowledge, no such singular integral equation has ever been solved. On the contrary, in the case of a real weight function $w(t)$, efficient methods for the numerical solution of singular integral equations, even with generalized kernels, have been proposed [3, 19, 20]. For Eq. (3) only some theoretical considerations contained in the book of Pogorzelski [24] may be proved applicable, but these have not led to a powerful numerical technique for the solution of this equation.

(iii) Singular integral equations of form (3) are very often encountered in practical applications. The authors, being familiar with plane elasticity problems, may note that complex singularities and generalized kernels are almost always present in elastic bodies of a complicated shape, wedge apices, cracks terminating on interfaces, etc. Unfortunately, although the singular integral equations for any such problem are known or may be easily obtained [3], nevertheless, their numerical solution was never tried. Thus, besides some special crack problems associated with complex singularities, which can be reduced to a singular integral equation of form (1), no other plane elasticity problems associated with complex singularities have been solved, or, in other words, no powerful method for the evaluation of generalized stress intensity factors associated with complex singularities, through the numerical solution of singular integral equations, is available.

(iv) Finally, a considerable difference between Eqs. (1) and (3), with respect to their numerical solution, is that the collocation points used for the numerical solution of Eq. (1) are the roots of appropriate polynomials, that is, the roots of analytic functions in the whole complex plane. The same does not, in general, hold for the numerical solution of Eq. (3). One more difference between Eqs. (1) and (3) is that Eq. (1) is reducible to an equivalent Fredholm integral equation, whereas Eq. (3) not.

3. Application of the Lobatto–Jacobi Numerical Integration Rule

In this section we will illustrate the proposed method for the numerical solution of Eq. (3) in the special case when we can apply the Lobatto–Jacobi numerical integration rule [21, 32, 33] for the approximation of the integrals. This means that we assume the integration interval $[a, b]$ to be finite. Then it can be readily reduced to the interval $[-1, 1]$, assumed to be the integration interval in the Lobatto–Jacobi numerical inegration rule, which has the form [32]

$$\int_{-1}^{1} w(t)\, g(t)\, dt = \sum_{i=1}^{n} A_i\, g(t_i) + E_n\,, \tag{4}$$

where the abscissas t_i and the weights A_i are determined as mentioned in Ref. [32], and the weight function $w(t)$ is of form (2). The characteristic feature of the Lobatto–Jacobi numerical integration rule is that it contains, among the abscissas used, the points $t = \pm 1$ (for this reason it is called a closed-type numerical integration rule). This property makes it very useful for the determination of stress intensity factors in plane elasticity [19, 33], or other analogous quantities (expressing the strength of singularities) in fluid mechanics or other branches of mathematical physics.

Although the Lobatto–Jacobi numerical integration rule was derived for the case of real singularities α and β [32] in the weight function $w(t)$, it remains also unaltered in the case of complex singularities, as proved in Ref. [21]. This means that the expressions giving the abscissas, the weights, and the error term E_n are the same in the cases of real and complex singularities. Moreover, the property of this rule to be exact ($E_n = 0$) for polynomials $g(t)$ of degree up to $(2n - 3)$ remains valid even in the case of complex singularities [21]. What is not easy to prove, is the convergence of this rule for increasing values of n to the correct value of the integral. Although the authors have not been able to prove that $E_n \to 0$ for $n \to \infty$, they may note that this seems to be true. In fact, it was seen that, for increasing values of n, the abscissas t_i used remain finite and lie, approximately, on a curve which, for $n \to \infty$, seems to tend to coincide with the interval $[-1, 1]$. This can be seen in the case when $\alpha = -0.5 + i1.0$ and $\beta = -0.5$ in Tables I and II (first columns), where the abscissas t_i for $n = 6$ and $n = 12$ are given. Unfortunately, the authors are unaware of any theorems or other results confirming the foregoing remark on the location (in the complex plane) of the abscissas t_i used. If this remark is proved correct (at least under some limitations on the values of α and β), then the results of Kahaner [34],

TABLE I

Abscissas t_i and collocation points x_k for the Lobatto–Jacobi numerical integration rule with $\alpha = -0.5 + i1.0$, $\beta = -0.5$, and $n = 6$

t_i	x_k
$1.00000 - i0.00000$	
	$0.96171 - i0.07457$
$0.80559 - i0.13201$	
	$0.57303 - i0.16135$
$0.28814 - i0.16361$	
	$-0.02167 - i0.14377$
$-0.32725 - i0.10953$	
	$-0.60018 - i0.06992$
$-0.81526 - i0.03387$	
	$-0.95274 - i0.00890$
$-1.00000 - i0.00000$	

TABLE II

Abscissas t_i and collocation points x_k for the Lobatto–Jacobi numerical integration rule with $\alpha = -0.5 + i1.0$, $\beta = -0.5$, and $n = 12$

t_i	x_k
$1.00000 - i0.00000$	
	$0.99382 - i0.01646$
$0.96214 - i0.03320$	
	$0.91067 - i0.04769$
$0.84077 - i0.05957$	
	$0.75391 - i0.06868$
$0.65188 - i0.07499$	
	$0.53677 - i0.07854$
$0.41090 - i0.07947$	
	$0.27684 - i0.07799$
$0.13728 - i0.07438$	
	$-0.00495 - i0.06869$
$-0.14698 - i0.06211$	
	$-0.28597 - i0.05423$
$-0.41910 - i0.04572$	
	$-0.54370 - i0.03702$
$-0.65727 - i0.02851$	
	$-0.75752 - i0.02060$
$-0.84244 - i0.01361$	
	$-0.91032 - i0.00784$
$-0.95981 - i0.00355$	
	$-0.98990 - i0.00090$
$-1.00000 - i0.00000$	

who proved the convergence of the equal-weight Chebyshev quadrature rule for large values of n (the abscissas used being in this case complex), may be used, in an analogous manner, in order that the convergence of the Lobatto–Jacobi quadrature rule be proved too.

In the case of Cauchy-type principal-value integrals, the Lobatto–Jacobi numerical integration rule (4) is modified as [33]

$$\int_{-1}^{1} w(t)\frac{g(t)}{t-x}\,dt = \sum_{i=1}^{n} A_i \frac{g(t_i)}{t_i - x} - g(x)\frac{q_n(x)}{\sigma_n(x)} + E_n\,, \qquad x \neq t_i \quad (i = 1, 2,..., n), \tag{5}$$

where [19, 33]

$$\sigma_n(x) = (1 - x^2)\frac{d}{dx} P_{n-1}^{(\alpha,\beta)}(x), \qquad q_n(x) = \int_{-1}^{1} w(t)\frac{\sigma_n(t)}{x-t}\,dt, \tag{6}$$

and $P_n^{(\alpha,\beta)}(x)$ denotes the classical Jacobi polynomial associated with the weight function $w(t)$ and of degree n.

Considering further the property of Jacobi polynomials [35, p. 170]

$$2\frac{d}{dx} P_n^{(\alpha,\beta)}(x) = (n + \alpha + \beta + 1)\, P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{7}$$

we can assume, slightly modifying the definitions of $\sigma_n(x)$ and $q_n(x)$, that

$$\sigma_n(x) = (1 - x^2)\, P_{n-2}^{(\alpha+1,\beta+1)}(x), \qquad q_n(x) = \Pi_{n-2}^{(\alpha+1,\beta+1)}(x). \tag{8}$$

In the second of Eqs. (8) the symbol $\Pi_n^{(\alpha,\beta}(z)$ denotes the function [36]

$$\Pi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1} w(t)\frac{P_n^{(\alpha,\beta)}(t)}{z-t}\,dt. \tag{9}$$

This function is a Jacobi function and is closely related to the Jacobi function of the second kind $Q_n^{(\alpha,\beta)}(z)$ [35, p. 170]. Also this function $\Pi_n^{(\alpha,\beta)}(z)$ was introduced by Elliott [36], who has also investigated its properties and developed asymptotic relations for its approximate evaluation. Of course, in the second of Eqs. (8) the function $\Pi_{n-2}^{(\alpha+1,\beta+1)}(x)$ denotes a Cauchy-type principal-value integral, which may also be evaluated as the mean value of the limiting values of $\Pi_{n-2}^{(\alpha+1,\beta+1)}(z)$ as $z \to x \pm 0i$, in accordance with the second Plemelj formula [22]. From the definition (9) of $\Pi_n^{(\alpha,\beta)}(z)$, it follows that this function presents a jump equal to $[-2\pi i w(x)\, P_n^{(\alpha,\beta)}(x)]$ at a point x of the interval $[-1, 1]$ when z crosses the real axis at the point x. This is a consequence of the first Plemelj formula [22]. Finally, $\Pi_n^{(\alpha,\beta)}(z)$ is analytic in the whole complex plane, the real interval $[-1, 1]$ deleted.

Now we will apply the Lobatto–Jacobi numerical integration rule to approximate the integrals in Eq. (3) (with $w(t)$ given by Eq. (2), $\alpha = -1$ and $\beta = 1$). Following the arguments of Ref. [21], we assume that the definitions of the known functions $A(x)$, $B(x)$, $K(t, x)$, and $f(x)$ can be extended to a sufficiently large domain G of the complex plane passing through the points (± 1) and surrounding the integration interval $[-1, 1]$. The usual form of this domain G may be seen in a paper by Donaldson and Elliott [37]. For analytic functions this extension of the domain of their definition is evident. For continuous functions along $(-1, 1)$, but not analytic in the complex plane, it might be necessary that these functions be approximated, e.g., through Bernstein polynomials, before the extension of the domain of their definition [21]. For convenience, even in the latter case, we will continue using the same symbols for these functions. Then, by applying the Lobatto–Jacobi numerical integration rule to the integrals of Eq. (3) and ignoring the error terms, which seems to be justified for sufficiently large values of n (excactly as in the case of real singularities [19]), we obtain

$$\left\{A(x)\,w(x) - B(x)\frac{q_n(x)}{\sigma_n(x)}\right\} g(x) + \sum_{i=1}^{n} A_i \left\{\frac{B(x)}{t_i - x} + K(t_i, x)\right\} g(t_i) \simeq f(x),$$

$$-1 < x < 1, \qquad x \neq t_i \quad (i = 1, 2, \ldots, n). \qquad (10)$$

In deriving Eq. (10) we have also implicitly assumed that the unknown function $g(t)$ is also analytic in a sufficiently large domain containing all abscissas t_i and surrounding the integration interval $[-1, 1]$.

Next, by taking into account the assumption made previously on the possibility of extension of the definition of the functions entering Eq. (10) and based on a theorem on the functions of a complex variable mentioned in Ref. [21], we can assume that Eq. (10) is approximately valid in the domain G defined previously. At this point we have to remark that the domain G used here passes through the points (± 1) and contains only the part $(-1, 1)$ of the real axis, which divides G into two subdomains. On the contrary, the domain G in Ref. [21] completely surrounded the interval $[-1, 1]$ and contained a part of the real axis wider than $[-1, 1]$.

We have also at this stage to give a proper definition of the function $q_n(x)$ in the domain G, which will be denoted in the sequel by $\tilde{q}_n(z)$. The definition of $q_n(x)$ given by the second of Eqs. (8) cannot be used outside $(-1, 1)$ since $q_n(z)$ is not analytic along $(-1, 1)$. This is due to the fact that $\Pi_{n-2}^{(\alpha+1,\beta+1)}(z)$ is defined in the Cauchy principal-value sense along $(-1, 1)$ as already stated. This must be taken into account, otherwise the results will be completely erroneous. This difficulty is overpassed by replacing $q_n(z)$ by the new function $\tilde{q}_n(z)$, defined in such a way that $\tilde{q}_n(z) \equiv q_n(z)$ for $z \in (-1, 1)$ and also that $\tilde{q}_n(z)$ be analytic along $(-1, 1)$, contrary to $q_n(z)$. Of course, then $\tilde{q}_n(z)$ will be not analytic along the remaining part of the real axis, but this is not of much importance since the abscissas and collocation points do not lie along this interval. Such an extension of the definition of a Cauchy-type principal-value integral outside the integration interval is used, to the authors' knowledge, for the first time in this paper and, perhaps, may be proved useful in other applications of Cauchy-type integrals too.

To be more specific, by taking into account that [33]

$$\Pi_n^{(\alpha,\beta)}(z) = 2(z-1)^\alpha (z+1)^\beta\, Q_n^{(\alpha,\beta)}(z), \qquad z \notin [-1, 1], \qquad (11)$$

as well as the property of Jacobi functions of the second kind $Q^{(\alpha,\beta)}(z)$ [35, p. 171]:

$$Q_n^{(\alpha,\beta)}(z) = -\frac{\pi}{2\sin\pi\alpha} P_n^{(\alpha,\beta)}(z) + 2^{\alpha+\beta-1}\frac{\Gamma(\alpha)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times (z-1)^{-\alpha}(z+1)^{-\beta} F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-z}{2}\right), \tag{12}$$

we find that

$$\Pi_n^{(\alpha,\beta)}(z) = -\frac{\pi}{\sin\pi\alpha}(z-1)^{\alpha}(z+1)^{\beta} P_n^{(\alpha,\beta)}(z) + 2^{\alpha+\beta}\frac{\Gamma(\alpha)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-z}{2}\right), \tag{13}$$

where the branches of $(z-1)^{\alpha}$ and $(z+1)^{\beta}$ are defined in such a way that these functions are analytic along the real axis with its $[-1, 1]$ excluded.

Furthermore, the function $\Pi_n^{(\alpha,\beta)}(z)$, defined in the principal-value sense, can be evaluated along the interval $(-1, 1)$, because of Eq. (13), as

$$\Pi_n^{(\alpha,\beta)}(x) = -\pi\cot\pi\alpha w(x)\, P_n^{(\alpha,\beta)}(x) + 2^{\alpha+\beta}\frac{\Gamma(\alpha)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-x}{2}\right). \tag{14}$$

This equation was obtained for the first time, but only for real values of α and β by Tricomi [38]. For complex values of α and β, but under the restriction that $(\alpha+\beta)$ is an integer, Eq. (14) was derived by Karpenko [39]. Nevertheless, the derivation of Eq. (14) based on Eq. (13) and, furthermore, on the properties of hypergeometric functions is free from restrictions on the values of α and β, besides the evident restriction that α is not a nonpositive integer.

Now we define the function

$$\tilde{\Pi}_n^{(\alpha,\beta)}(z) = -\pi\cot\pi\alpha(1-z)^{\alpha}(1+z)^{\beta} P_n^{(\alpha,\beta)}(z) + 2^{\alpha+\beta}\frac{\Gamma(\alpha)\,\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)} \times F\left(n+1, -n-\alpha-\beta; 1-\alpha; \frac{1-z}{2}\right). \tag{15}$$

This function is defined in such a way that it coincides with $\Pi_n^{(\alpha,\beta)}(x)$ for z lying on $(-1, 1)$. It is also analytic in the whole plane, the real intervals $(-\infty, -1]$ and $[1, \infty)$ excluded. This function is the continuation of $\Pi_n^{(\alpha,\beta)}(x)$ outside $(-1, 1)$, not $\Pi_n^{(\alpha,\beta)}(z)$. Thus, in Eq. (10), when we consider it valid outside the interval $(-1, 1)$ and inside the domain G mentioned previously, we have to consider the function $q_n(z)$ defined by the second of Eqs. (8), but with $\Pi_{n-2}^{(\alpha+1,\beta+1)}(z)$ replaced by $\tilde{\Pi}_{n-2}^{(\alpha+1,\beta+1)}(z)$, defined with the help of Eq. (15). Then $q_n(z)$ will be denoted by $\tilde{q}_n(z)$. Otherwise, the results will not be correct.

Next, following the developments of Refs. [19, 20], we apply Eq. (10) at a certain number (in fact $(n-1)$) of collocation points x_k, defined as the roots of the transcendental equation

$$A(z)\, w(z)\, \sigma_n(z) - B(z)\, \tilde{q}_n(z) = 0, \qquad w(z) = (1-z)^{\alpha}(1+z)^{\beta}. \tag{16}$$

The arguments of Refs. [19, 20] assure the existence of these collocation points, which interchange with the abscissas used for real values of α and β ($\alpha, \beta > -1$). For complex values of α and β (Re α, Re $\beta > -1$), although no proof of this assertion seems possible, yet the situation is not much different. In Tables I and II the abscissas and collocation points used for $\alpha = -0.5 + i1.0$, $\beta = -0.5$, $n = 6$ and 12, respectively, and $A(z) \equiv 0$, are seen to be alternating along a line in the complex plane connecting the end-points $(+1)$ and (-1) of the interval $[-1, 1]$. This line for real α and β coincides with the interval $[-1, 1]$. As was also previously mentioned, this line seems to tend to coincide with the interval $[-1, 1]$, even in the case of complex singularities, but when a sufficiently large number of abscissas n is used.

Finally, by using the complex collocation points x_k, as already defined, we can reduce Eq. (10), and moreover Eq. (3) for $a = -1, b = 1$ and $w(t)$ defined by Eq. (2), to the following system of linear equations:

$$\sum_{i=1}^{n} A_i \left\{ \frac{B(x_k)}{t_i - x_k} + K(t_i, x_k) \right\} g(t_i) \simeq f(x_k), \qquad k = 1, 2, \ldots, n-1. \tag{17}$$

This system of linear equations should be supplemented by one more linear equation, easily resulting, as in the case of real singularities α and β, by using a condition resulting from physical considerations. We will not enter into any more details and comments, restricting ourselves to refer to the appropriate references [3, 19–24, 28, 29, 31–33, 39–40].

4. The General Case

The arguments of the previous section can be readily generalized to the general case of Eq. (3) under the assumptions made in Section 2. The basic steps in solving such an equation are:

(i) To properly select the numerical integration rule to be used. Perhaps, this will be a well-known numerical integration rule, like the Lobatto–Jacobi rule, considered in Section 3, or the Gauss–Jacobi rule, already applied to the numerical solution of Cauchy-type singular integral equations by the present authors [21, 40]. In general, even when no appropriate numerical integration rule for the approximation of the integrals in Eq. (3) is available, one can derive such a rule for an arbitrary weight function $w(t)$ and apply it to the numerical evaluation of both regular [37] and Cauchy-type principal-value integrals [41]. Also the functions $q_n(z)$ and $\tilde{q}_n(z)$ corresponding to this rule can be easily derived.

(ii) To evaluate the generally complex collocation points x_k to be used, through the solution of a transcendental equation of form (16). It must be noted at this point that the extension of the definition of $q_n(x)$ outside the interval (a, b), $\tilde{q}_n(z)$, as made in Section 3, and in such a way that it is analytic in the domain G (surrounded by a contour passing through a and b) is not a difficult matter and can be easily achieved both for complex power and for logarithmic singularities. To do this, we have to properly select the branch of the weight function $w(z)$ in the complex plane so that it does not present any jump across the interval (a, b) but only in the part of the real axis outside this interval. Of course, these arguments hold only under the initially made assumption that the singularities of the weight function are restricted in the neighborhoods of the end-points a and b of the integration interval $[a, b]$.

(iii) After the evaluation of the collocation points, the resulting system of linear equations of form (17) has to be solved, probably supplemented by one more linear equation, as mentioned in Section 3. The values of the unknown function $g(t)$ at the abscissas used t_i will be determined from the solution of the system of linear equations.

(iv) Finally, an interpolation procedure may be used for the approximation of $g(t)$ through a polynomial series so that its determination along the whole interval $[a, b]$ or, perhaps, even outside it, is possible.

For a more detailed analysis of these steps in the case of a real weight function $w(t)$, one may see Refs. [19, 20]. Some comments on the convergence of the technique to the correct results, although no proof of it, can be found in Ref. [21]

5. An Application

As an application we consider the following well-known singular integral equation, already considered by several authors (see e.g., [42–44]):

$$\frac{1}{\pi}\int_0^1 w(t)\left\{\frac{1}{t-x}+\frac{\lambda}{t+x}\right\} g(t)\,dt = 1, \qquad 0 < x < 1, \tag{18}$$

under the additional condition

$$\int_0^1 w(t)\,g(t)\,dt = 0. \tag{19}$$

The weight function $w(t)$ is of the form

$$w(t) = (1-t)^{-1/2}\,t^{\alpha}, \tag{20}$$

where α is determined by

$$\cos \pi\alpha = -\lambda, \qquad -1 < \operatorname{Re} \alpha < 0. \tag{21}$$

Equation (18), when λ is real, results in the antiplane elasticity problem of a crack terminating perpendicularly at a bimaterial interface. Furthermore, this equation has a closed-form solution [42–44]. Since the authors are not aware of any other singular integral equation of the type considered in this paper having a closed-form solution, they have used Eq. (18) to check their theoretical considerations of the previous sections, assuming that the constant λ is a complex number. Then α results from Eq. (21) as complex too. At this stage it can be remarked that, although the solution given by Bueckner [42] was obtained under the assumption that λ is a real number, the method of solution used by Barnett [43], although considered for real values of λ too, seems to be also applicable, even when λ is a complex number.

Equation (18) was solved by using the Lobatto–Jacobi method, as proposed in Section 3, for various values of α, the corresponding values of λ determined from Eq. (21), after a transformation of the integration interval [0, 1] so that it coincides with [−1, 1]. The values of the "stress intensity factors" $K(0)$ and $K(1)$, proportional to the values of $g(0)$ and $g(1)$ of the unknown function $g(t)$ at $t = 0$ and $t = 1$, respectively [44] (the ratio μ_2/μ_1 in Ref. [44] considered here equal to 1 for the evaluation of $K(0)$), were evaluated for $n = 3(3)15$ and their convergence to their theoretical values given by [44]

$$K(0) = -2^{-\alpha-1/2}(1+\alpha)/\sin(\tfrac{1}{2}\pi\alpha), \qquad K(1) = \alpha/\sin(\tfrac{1}{2}\pi\alpha), \tag{22}$$

obtained from the closed-form solution of Eq. (18), are shown in Table III for $\alpha = -0.5 + i1.0$ and $\alpha = -0.7 + i1.0$ ($\beta = -0.5$). The results obtained are good enough although the imaginary part of α is sufficiently large. Usually, in plane elasticity problems the imaginary part of a complex singularity is much less than its real part. Furthermore, for a value of α in Eq. (18) such that $0 < -\mathrm{Re}\,\alpha < 0.5$, the developments of Ref. [44] for real singularities should be taken into account in the

TABLE III

Convergence of the numerically obtained values of the "generalized stress intensity factors" in the solution of Eq. (18) to their theoretical values for an increasing number n of the abscissas used

	$\alpha = -0.5 + i1.0, \beta = -0.5$		$\alpha = -0.7 + i1.0, \beta = -0.5$	
n	$K(0)$	$K(1)$	$K(0)$	$K(1)$
3	$-0.10836 + i0.39784$	$0.45413 - i0.17651$	$0.08030 + i0.34246$	$0.44840 - i0.24749$
6	$0.26660 + i0.49186$	$0.43382 - i0.16593$	$0.26561 + i0.44494$	$0.42885 - i0.24714$
9	$0.14414 + i0.44638$	$0.43385 - i0.16574$	$0.24280 + i0.41247$	$0.42852 - i0.24706$
12	$0.16846 + i0.40401$	$0.43382 - i0.16573$	$0.25256 + i0.40956$	$0.42851 - i0.24701$
15	$0.19139 + i0.40744$	$0.43382 - i0.16574$	$0.25476 + i0.41295$	$0.42853 - i0.24700$
Theoretical value	$0.18706 + i0.42506$	$0.43382 - i0.16574$	$0.25296 + i0.41495$	$0.42854 - i0.24703$

case of complex singularities too. Finally, a considerable part of the errors in the results of Table III is simply due to ignoring the pole of the Fredholm kernel in Eq. (18) in the interval $(-1, 0)$. This pole can of course be taken into account as proposed in Ref. [45].

Finally, some remarks of computational character may be added:

(i) In evaluating the abscissas t_i and the collocation points x_k, the Newton–Raphson method, particularly convenient for the classical orthogonal polynomials and their associated functions even in the case of complex roots, has been used. The simple trigonometric and real abscissas and collocation points of the Lobatto–Chebyschev method [46] (with the same n) have been used as first approximations. Of course, this technique is not applicable for sufficiently large values of $\operatorname{Im}\alpha$ or $\operatorname{Im}\beta$.

(ii) The property of the Jacobi functions $\Pi_n^{(\alpha,\beta)}(z)$ that [36]

$$\Pi_n^{(\alpha,\beta)}(-z) = (-1)^{n+1}\Pi_n^{(\beta,\alpha)}(z) \tag{23}$$

has been taken into account, together with Eq. (14), for the development of a formula analogous to (15) for the computation of $\tilde{\Pi}_n^{(\alpha,\beta)}(z)$ for $\operatorname{Re} z < 0$. Equation (15) was used for the evaluation of $\tilde{\Pi}_n^{(\alpha,\beta)}(z)$ for $\operatorname{Re} z > 0$. Thus, a rapid convergence of the hypergeometric function series [47, p. 556] was achieved.

(iii) The evaluation of the values of the gamma function for complex arguments, required in Eq. (15), was based on tables included in Ref. [47, pp. 277–287].

References

1. N. I. Ioakimidis and P. S. Theocaris, *Strojn. Čas.* **29** (1978), 448–456.
2. P. S. Theocaris and N. I. Ioakimidis, *Quart. J. Mech. Appl. Math.* **30** (1977), 437–448.
3. N. I. Ioakimidis, "General Methods for the Solution of Crack Problems in the Theory of Plane Elasticity," (in Greek), Doctoral thesis, The National Technical University of Athens, Athens, 1976 (University Microfilms Order No. 76-21,056).
4. M. P. Savruk, *J. Appl. Math. Mech.* **39** (1975), 685–692 [Translation of *Prikl. Mat. Mekh.* **39** (1975), 717–723].
5. B. C. Sheu, "Numerical Solution of Singular Integral Equations, with Applications to Elastic Wave Propagation Problems," Ph.D. Thesis, Northwestern University, Evanston, Ill., 1976 (University Microfilms Order No. 77-1351).
6. M. Comninou and J. Dundurs, *Proc. Roy. Soc. London Ser. A* **356** (1977), 509–528.
7. B. L. Karihaloo, "Fracture of Solids Containing Arrays of Cracks," The University of Newcastle (Faculty of Engineering), Newcastle, Australia, Engineering Bulletin CE7, 1977.
8. B. L. Karihaloo, *Internat. J. Solids Struct.* **13** (1977), 367–375.
9. S. G. Mikhlin, "Integral Equations," Pergamon, Oxford, 1957.
10. S. R. Majumdar, *Z. Angew. Math. Mech.* **54** (1974), 515–517.
11. C. T. Tai, *J. Appl. Phys.* **23** (1952), 909–916.
12. B. Rulf, *IEEE Trans. Microwave Theory Tech.* MTT-**24** (1976), 539–541.
13. B. Rulf and N. Kedem, *J. Sound Vibr.* **45** (1976), 15–28.
14. B. Rulf, *J. Opt. Soc. Amer.* **65** (1975), 1248–1252.
15. J. A. DeSanto and O. Shisha *J. Computational Phys.* **15** (1974), 286–292.

16. V. V. IVANOV, "The Theory of Approximate Methods and Their Application to the Numerical Solution of Singular Integral Equations," Noordhoff, Leyden, the Netherlands, 1976.
17. M. L. DOW AND D. ELLIOTT, "The Numerical Solution of Singular Integral Equations over $(-1, 1)$," The University of Tasmania, Hobart, Tasmania, Technical Report No. 91, 1977.
18. M. L. DOW, "Singular Integral Equations," Ph.D. thesis, The University of Tasmania, Hobart, Tasmania, 1977.
19. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *Trans. Acad. Athens* **40** (1977), 1–39.
20. P. S. THEOCARIS, *Serdica, Bulg. Math. Publ.* **2** (1976), 252–275.
21. P. S. THEOCARIS AND N. I. IOAKIMIDIS *Z. Angew. Math. Phys.* **28** (1977), 1085–1098.
22. N. I. MUSKHELISHVILI, "Singular Integral Equations," Noordhoff, Leyden, the Netherlands, 1977.
23. F. D. GAKHOV, "Boundary Value Problems," Pergamon and Addison–Wesley, Oxford, 1966.
24. W. POGORZELSKI, "Integral Equations and Their Applications," Vol. 1, Pergamon and PWN—Polish Scientific Publishers, Warszawa, 1966.
25. P. S. THEOCARIS, *Internat. J. Engrg. Sci.* **12** (1974), 107–120.
26. N. I. IOAKIMIDIS, "Wedge and Crack Problems in Plane Elasticity," Master thesis, The National Technical University of Athens, 1973.
27. W. E. WARREN, *Z. Angew. Math. Phys.* **28** (1977), 177–181.
28. F. ERDOGAN, G. D. GUPTA, AND T. S. COOK, Numerical solution of singular integral equations, *in* "Mechanics of Fracture, Vol. 1, Methods of Analysis and Solutions of Crack Problems," (G. C. Sih, Ed.), Chap. 7, pp. 368–425, Noordhoff, Leyden, the Netherlands, 1973.
29. F. ERDOGAN, Complex function technique, *in* "Continuum Physics, Vol. 2, Continuum Mechanics of Single-Substance Materials," (A. C. Eringen, Ed.), Part III, Chap. 3, pp. 523–603, Academic Press, New York, 1975.
30. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Internat. J. Fracture* **12** (1976), 921–922.
31. N. P. VEKUA, "Systems of Singular Integral Equations," Noordhoff, Groningen, the Netherlands, 1967.
32. Z. KOPAL, "Numerical Analysis," Chapman and Hall, London, 1961.
33. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Apl. Mat.* **23** (1978), 439–452.
34. D. K. KAHANER, *Math. Comp.* **25** (1971), 827–830.
35. "Higher Transcendental Functions," (A. Erdélyi, Ed.), Vol. II, McGraw–Hill, New York, 1953.
36. D. ELLIOTT, *Math. Comp.* **25** (1971), 309–315.
37. J. D. DONALDSON AND D. ELLIOTT, *SIAM J. Numer. Anal.* **9** (1972), 573–602.
38. F. G. TRICOMI, *Quart. J. Math., Oxford Ser. 2*, **2** (1951), 199–211.
39. L. N. KARPENKO, *J. Appl. Math. Mech.* **30** (1966), 668–675 [Translation of *Prikl. Mat. Mekh.* **30** (1966), 564–569].
40. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *Bull. Calcutta Math. Soc.* to appear.
41. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Rev. Roumaine Sci. Tech. Sér. Méc. Appl.* **22** (1977), 803–818.
42. H. F. BUECKNER, *J. Math. Anal. Appl.* **14** (1966), 392–426.
43. D. M. BARNETT, *Acta Metall.* **15** (1967), 589–594.
44. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *Internat. J. Fracture* **13** (1977), 549–552.
45. N. I. IOAKIMIDIS AND P. S. THEOCARIS, *Internat. J. Fracture* **14** (1978), 469–484.
46. P. S. THEOCARIS AND N. I. IOAKIMIDIS, *Quart. Appl. Math.* **35** (1977), 173–183.
47. "Handbook of Mathematical Functions," (M. Abramowitz and I. A. Stegun, Eds), Dover, New York, 1965.

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